

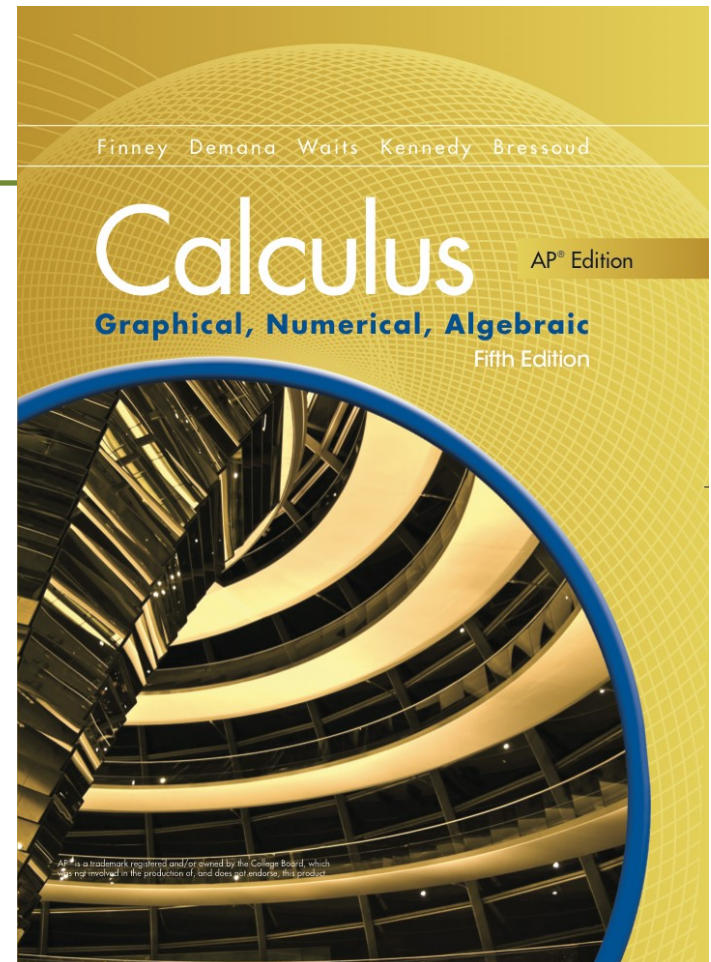
Chapter 2

Limits and Continuity



Section 2.1

Rates of Change and Limits



Quick Review

In Exercises 1–4, find $f(2)$.

1. $f(x) = 2x^3 - 5x^2 + 4$

2. $f(x) = \frac{4x^2 - 5}{x^3 + 4}$

3. $f(x) = \sin\left(\pi \frac{x}{2}\right)$

4. $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \frac{1}{x^2 - 1}, & x \geq 2 \end{cases}$

Quick Review

In Exercises 5 – 8, write the inequality in the form $a < x < b$.

5. $|x| < 4$

6. $|x| < c^2$

7. $|x - 2| < 3$

8. $|x - d| < d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\frac{x^2 - 3x - 18}{x + 3}$

10. $\frac{2x^2 - x}{2x^2 + x - 1}$

Quick Review Solutions

In Exercises 1–4, find $f(2)$.

$$1. \quad f(x) = 2x^3 - 5x^2 + 4 \quad 0 \qquad 2. \quad f(x) = \frac{4x^2 - 5}{x^3 + 4} \quad \frac{11}{12}$$

$$3. \quad f(x) = \sin\left(\pi \frac{x}{2}\right) \quad 0$$

$$4. \quad f(x) = \begin{cases} 3x - 1, & x < 2 \\ \frac{1}{x^2 - 1}, & x \geq 2 \end{cases} \quad \frac{1}{3}$$

Quick Review Solutions

In Exercises 5 – 8, write the inequality in the form $a < x < b$.

5. $|x| < 4$ - $4 < x < 4$

6. $|x| < c^2$ - $c^2 < x < c^2$

7. $|x - 2| < 3$ - $1 < x < 5$

8. $|x - c| < d^2$ - $c - d^2 < x < c + d^2$

In Exercises 9 and 10, write the fraction in reduced form.

9. $\frac{x^2 - 3x - 18}{x + 3}$ - $x - 6$

10. $\frac{2x^2 - x}{2x^2 + x - 1}$ - $\frac{x}{x + 1}$

What you'll learn about

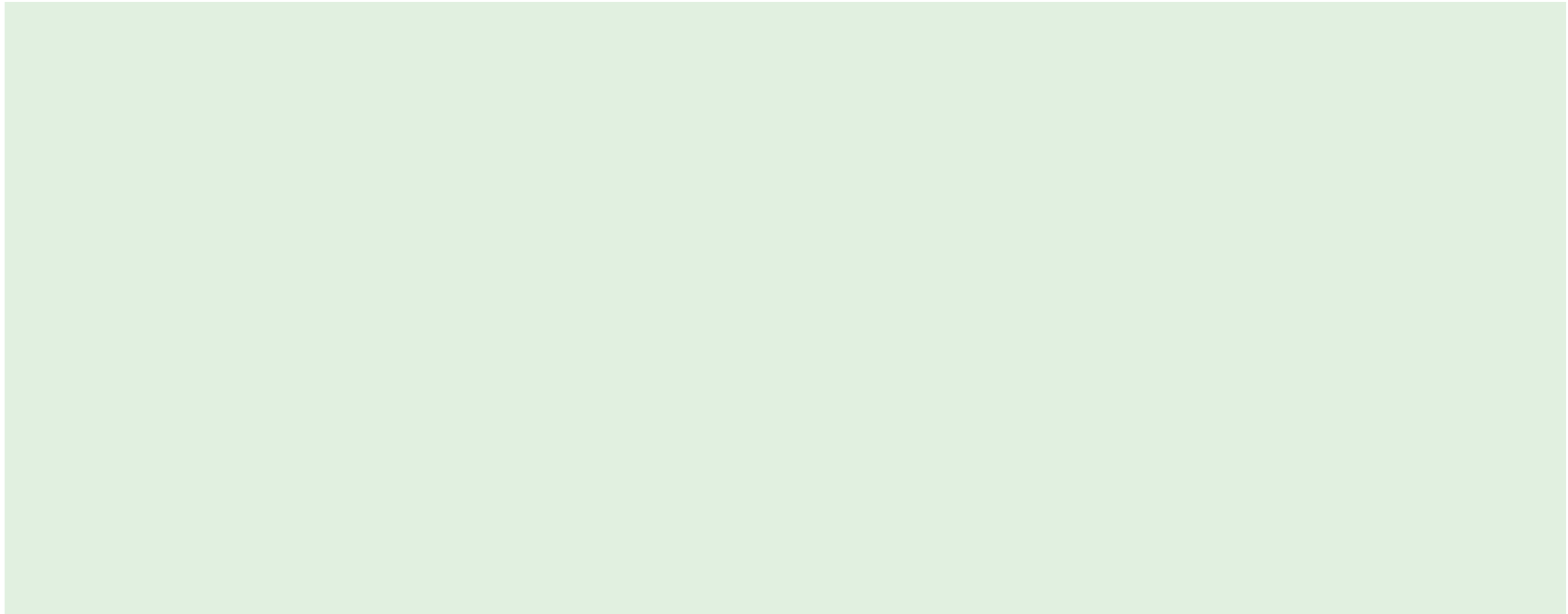
- Interpretation and expression of limits using correct notation
- Estimation of limits using numerical and graphical information
- Limits of sums, differences, products, quotients, and composite functions
- Interpretation and expression of one-sided limits
- The Squeeze Theorem

...and why

Limits can be used to describe continuity, the derivative

and the integral: the ideas giving the foundation of calculus.

Average and Instantaneous Speed



A body's average speed during an interval of time is found by dividing the distance covered by the elapsed time.

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

Average and Instantaneous Speed

The **average speed** or **average rate of change** of a moving body during an interval of time is found by dividing the change in distance or position by the change in time.

The speed of a falling rock is always increasing. If we know the position as a function of time, we can calculate average speed over any given interval of time. But we can also talk about its **instantaneous speed** or **instantaneous rate of change**, the speed at one instant of time. As we will

Definition of Limit

Let c and L be real numbers. The function f has limit L as x approaches c if, given any positive number ε , there is a positive number δ such that for all x ,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \rightarrow c} f(x) = L$$

Definition of Limit continued

The sentence $\lim_{x \rightarrow c} f(x) = L$ is read, “The limit of f of x as x approaches c equals L ”. The notation means that the values $f(x)$ of the function f approach or equal L as the values of x approach (but do not equal) c .

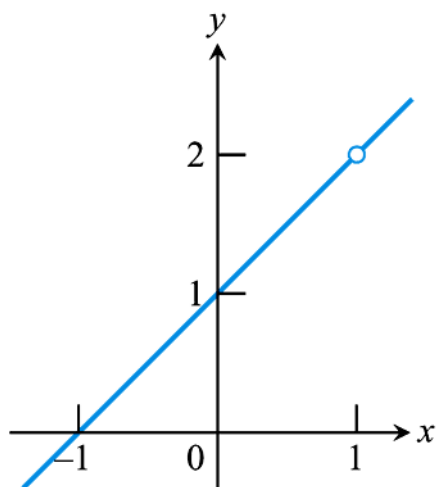
Figure 2.2 illustrates the fact that the existence of a limit as $x \rightarrow c$ never depends on how the function may or may not be defined at c .

Definition of Limit continued

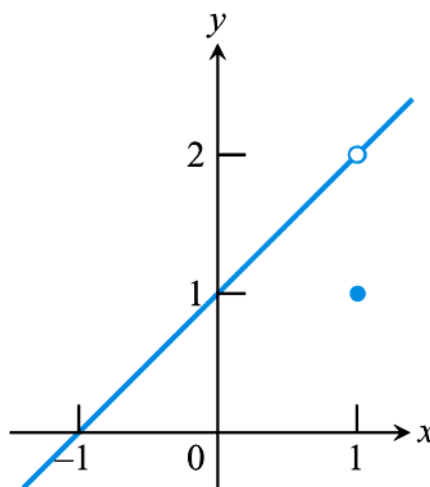
The function f has limit 2 as $x \rightarrow 1$ even though f is not defined at 1.

The function g has limit 2 as $x \rightarrow 1$ even though $g(1) \neq 2$.

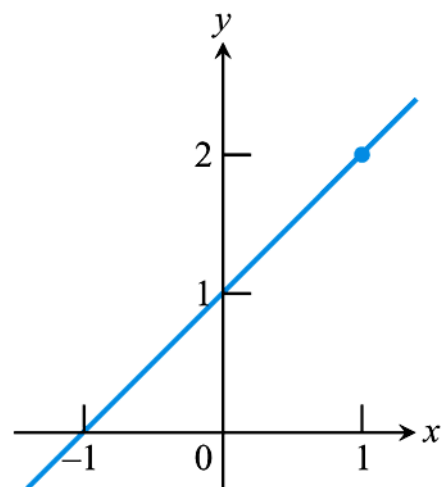
The function h is the only one whose limit as $x \rightarrow 1$ equals its value at $x=1$.



(a) $f(x) = \frac{x^2 - 1}{x - 1}$



(b) $g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$



(c) $h(x) = x + 1$

Properties of Limits

If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \text{ then}$$

1. *Sum Rule:*
$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:*
$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

Properties of Limits continued

3. *Product Rule* $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

The limit of the product of two functions is the product of their limits.

4. *Constant Multiple Rule* $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule* $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

Properties of Limits continued

6. *Power Rule* If r and s are integers, $s \neq 0$, then

$$\lim_{x \rightarrow c} (f(x))^{\frac{r}{s}} = L^{\frac{r}{s}}$$

provided that $L^{\frac{r}{s}}$ is a real number.

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Other properties of limits:

$$\lim_{x \rightarrow c} (k) = k$$

$$\lim_{x \rightarrow c} (x) = c$$

Example Properties of Limits

Use any of the properties of limits to find

$$\lim_{x \rightarrow c} (3x^3 + 2x - 9)$$

$$\begin{aligned} \lim_{x \rightarrow c} (3x^3 + 2x - 9) &= \lim_{x \rightarrow c} 3x^3 + \lim_{x \rightarrow c} 2x - \lim_{x \rightarrow c} 9 && \text{sum and difference rules} \\ &= 3c^3 + 2c - 9 && \text{product and multiple rules} \end{aligned}$$

Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \rightarrow c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \text{ provided that } g(c) \neq 0.$$

Example Limits

Use Theorem 2 to find $\lim_{x \rightarrow 5} (4x^2 - 2x + 6)$

$$\lim_{x \rightarrow 5} (4x^2 - 2x + 6) = 4(5)^2 - 2(5) + 6$$

$$= 4(25) - 10 + 6$$

$$= 100 - 10 + 6$$

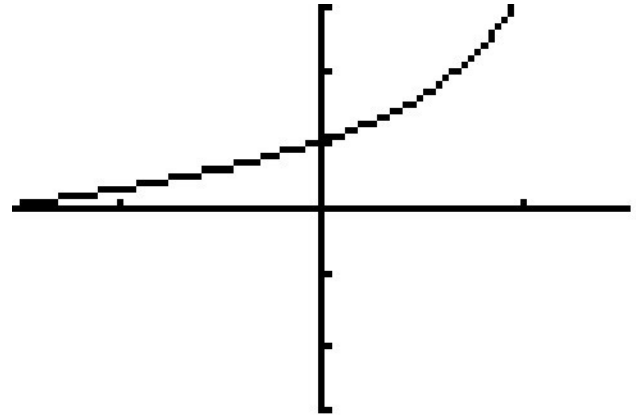
$$= 96$$

Evaluating Limits

As with polynomials, limits of many familiar functions can be found by substitution at points where they are defined. This includes trigonometric functions, exponential and logarithmic functions, and composites of these functions.

Example Limits

Find $\lim_{x \rightarrow 0} \frac{1 + \sin x}{\cos x}$



Solve graphically:

The graph of $f(x) = \frac{1 + \sin x}{\cos x}$ suggests that the limit exists and is 1.

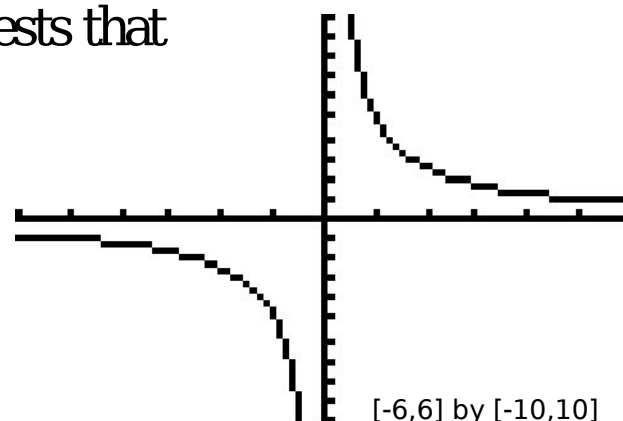
Confirm Analytically:

$$\begin{aligned} \text{Find } \lim_{x \rightarrow 0} \frac{1 + \sin x}{\cos x} &= \frac{\lim_{x \rightarrow 0} (1 + \sin x)}{\lim_{x \rightarrow 0} \cos x} = \frac{(1 + \sin 0)}{\cos 0} \\ &= \frac{1 + 0}{1} = 1 \end{aligned}$$

Example Limits

Find $\lim_{x \rightarrow 0} \frac{5}{x}$

Solve graphically: The graph of $f(x) = \frac{5}{x}$ suggests that the limit does not exist.



Confirm Analytically:

We can't use substitution in this example because when x is replaced by 0, the denominator becomes 0 and the function is undefined.

This would suggest that we rely on the graph to see that the limit does not exist.

One-Sided and Two-Sided Limits

Sometimes the values of a function f tend to different limits as x approaches a number c from opposite sides. When this happens, we call the limit of f as x approaches c from the right the right-hand limit of f at c and the limit as x approaches c from the left the left-hand limit.

right-hand: $\lim_{x \rightarrow c^+} f(x)$ The limit of f as x approaches c from the right.

left-hand: $\lim_{x \rightarrow c^-} f(x)$ The limit of f as x approaches c from the left.

One-Sided and Two-Sided Limits continued

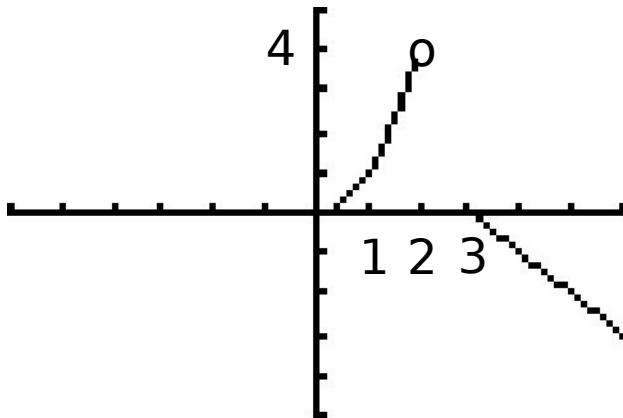
We sometimes call $\lim_{x \rightarrow c} f(x)$ the two-sided limit of f at c to distinguish it from the one-sided right-hand and left-hand limits of f at c .

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^+} f(x) = L \text{ and } \lim_{x \rightarrow c^-} f(x) = L.$$

Example One-Sided and Two-Sided Limits

Find the following limits from the given graph.



- a. $\lim_{x \rightarrow 0^+} f(x) = 0$
- b. $\lim_{x \rightarrow 2^+} f(x) = \text{Does Not Exist}$
- c. $\lim_{x \rightarrow 2^-} f(x) = 4$
- d. $\lim_{x \rightarrow 2} f(x) = \text{Does Not Exist}$
- e. $\lim_{x \rightarrow 3^+} f(x) = 0$

Squeeze Theorem

If we cannot find a limit directly, we may be able to find it indirectly with the Squeeze Theorem. The theorem refers to a function f whose values are squeezed between the values of two other functions, g and h .

If g and h have the same limit as $x \rightarrow c$ then f has that limit too.

Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c , and

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L,$$

then

$$\lim_{x \rightarrow c} f(x) = L$$